Kinetic Equations

Solution to the Exercises

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Recall the Vlasov Equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E\left(t, x\right) := -\nabla\left(\frac{1}{|x|} * \rho_t\right)\left(x\right), \\ \rho_t\left(x\right) := \int_{\mathbb{R}^3} f\left(t, x, v\right) dv, \\ f\left(0, x, v\right) = f_0\left(x, v\right). \end{cases} \tag{1}$$

Exercise 1

Let $f \in C^1_{\mathrm{c}}\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$. Prove that f satisfies

$$\operatorname*{ess\,sup}_{\substack{y \in B_R(x), \\ w \in B_R(v)}} f\left(y + tv, w\right) \in L^\infty\left(\left(0, T\right) \times \mathbb{R}^3_x; L^1\left(\mathbb{R}^3_v\right)\right), \tag{2}$$

$$\operatorname*{ess\,sup}_{\substack{y \in B_R(x), \\ w \in B_R(v)}} \left|\nabla f\left(y + tv, w\right)\right| \in L^\infty\left(\left(0, T\right) \times \mathbb{R}^3_x; L^1\left(\mathbb{R}^3_v\right) \cap L^2\left(\mathbb{R}^3_v\right)\right). \tag{3}$$

Proof. Notice that if $f \in C^1_{\mathrm{c}}\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$, then there exists a constant M such that if we denote $\chi_M := \chi_{B_M(0)}$ we get

$$\left|f\left(x, v\right)\right| \leqslant \|f\|_\infty \chi_M\left(x\right) \chi_M\left(v\right), \tag{4}$$

$$\left|\nabla f\left(x, v\right)\right| \leqslant \|\nabla f\|_\infty \chi_M\left(x\right) \chi_M\left(v\right). \tag{5}$$

In particular this implies that

$$\left|f\left(y + tv, w\right)\right| \leqslant \|f\|_\infty \chi_M\left(y + tv\right) \chi_M\left(w\right), \tag{6}$$

$$\left|\nabla f\left(y + tv, w\right)\right| \leqslant \|\nabla f\|_\infty \chi_M\left(y + tv\right) \chi_M\left(w\right). \tag{7}$$

If $w \in B_R\left(v\right)$ and $w \in B_M\left(0\right)$, then $v \in B_{R+M}\left(0\right)$, and therefore, for any $w \in B_R\left(v\right)$ we get

$$\left|f\left(y + tv, w\right)\right| \leqslant \|f\|_\infty \chi_M\left(y + tv\right) \chi_{R+M}\left(v\right), \tag{8}$$

$$\left|\nabla f\left(y + tv, w\right)\right| \leqslant \|\nabla f\|_\infty \chi_M\left(y + tv\right) \chi_{R+M}\left(v\right). \tag{9}$$

On the other hand we have $|x| \leqslant |x - y| + |y + tv| + |tv|$, therefore if $y \in B_R\left(x\right)$, $w \in B_R\left(v\right)$, $w \in B_M\left(0\right)$ and $y + tv \in B_M\left(0\right)$ we get $x \in B_{(1+t)(R+M)}\left(0\right)$.

As a consequence, for any $y \in B_R\left(x\right)$ and $w \in B_R\left(v\right)$ we have

$$\left|f\left(y + tv, w\right)\right| \leqslant \|f\|_\infty \chi_{(1+t)(R+M)}\left(x\right) \chi_{R+M}\left(v\right), \tag{10}$$

$$\left|\nabla f\left(y + tv, w\right)\right| \leqslant \|\nabla f\|_\infty \chi_{(1+t)(R+M)}\left(x\right) \chi_{R+M}\left(v\right). \tag{11}$$

This allows us to conclude

$$\operatorname*{ess\,sup}_{(t,x)\in(0,T)\times\mathbb{R}^3_x} \int_{\mathbb{R}^3} \left| \operatorname*{ess\,sup}_{\substack{y\in B_R(x),\\ w\in B_R(v)}} f\left(y+tv,w\right) \right| dv \leqslant \tag{12}$$

$$\leqslant \operatorname*{ess\,sup}_{(t,x)\in(0,T)\times\mathbb{R}^3_x} \|f\|_\infty \int_{\mathbb{R}^3} \chi_{(1+t)(R+M)}\left(x\right)\chi_{R+M}\left(v\right) dv \tag{13}$$

$$\leqslant \frac{4}{3}\pi \|f\|_\infty \left(R+M\right)^3, \tag{14}$$

$$\operatorname*{ess\,sup}_{(t,x)\in(0,T)\times\mathbb{R}^3_x} \int_{\mathbb{R}^3} \left| \operatorname*{ess\,sup}_{\substack{y\in B_R(x),\\ w\in B_R(v)}} \left|\nabla f\left(y+tv,w\right)\right| \right| dv \leqslant \tag{15}$$

$$\leqslant \operatorname*{ess\,sup}_{(t,x)\in(0,T)\times\mathbb{R}^3_x} \|\nabla f\|_\infty \int_{\mathbb{R}^3} \chi_{(1+t)(R+M)}\left(x\right)\chi_{R+M}\left(v\right) dv \tag{16}$$

$$\leqslant \frac{4}{3}\pi \|\nabla f\|_\infty \left(R+M\right)^3, \tag{17}$$

$$\operatorname*{ess\,sup}_{(t,x)\in(0,T)\times\mathbb{R}^3_x} \left(\int_{\mathbb{R}^3} \left| \operatorname*{ess\,sup}_{\substack{y\in B_R(x),\\ w\in B_R(v)}} \left|\nabla f\left(y+tv,w\right)\right| \right|^2 dv \right)^{\frac{1}{2}} \leqslant \tag{18}$$

$$\leqslant \operatorname*{ess\,sup}_{(t,x)\in(0,T)\times\mathbb{R}^3_x} \|\nabla f\|_\infty \left(\int_{\mathbb{R}^3} \chi_{(1+t)(R+M)}\left(x\right)\chi_{R+M}\left(v\right) dv \right)^{\frac{1}{2}} \tag{19}$$

$$\leqslant \sqrt{\frac{4}{3}\pi} \|\nabla f\|_\infty \left(R+M\right)^{\frac{3}{2}}. \tag{20}$$

□

Exercise 2

Let f be a solution of the Vlasov equation (1) such that $\rho \in L^\infty\left((0,T);L^1\left(\mathbb{R}^3\right)\cap L^\infty\left(\mathbb{R}^3\right)\right)$ and with f_0 such that

$$\operatorname*{ess\,sup}_{\substack{y\in B_R(x),\\ w\in B_R(v)}} f_0\left(y+tv,w\right) \in L^\infty\left((0,T)\times\mathbb{R}^3_x;L^1\left(\mathbb{R}^3_v\right)\right), \tag{21}$$

$$\operatorname*{ess\,sup}_{\substack{y\in B_R(x),\\ w\in B_R(v)}} \left|\nabla f_0\left(y+tv,w\right)\right| \in L^\infty\left((0,T)\times\mathbb{R}^3_x;L^1\left(\mathbb{R}^3_v\right)\cap L^2\left(\mathbb{R}^3_v\right)\right). \tag{22}$$

Prove that

$$\nabla_v f\left(t,x,v\right) \in L^\infty\left((0,T)\times\mathbb{R}^3_x;L^2\left(\mathbb{R}^3_v\right)\right) \tag{23}$$

Hint: Recall that we saw in class that under this hypotheses there exists $\alpha\in(0,1)$ such that $E\in L^\infty\left((0,T);C^{1,\alpha}\left(\mathbb{R}^3\right)\right)$ and use the explicit formula for f given through the characteristics of the problem.

Proof. Recall that if we define $X(s,t,z)$, $V(s,t,z)$ as the characteristics associated to the Vlasov equation such that

$$\begin{cases} \partial_s X(s,t,z) = V(s,t,z), \\ \partial_s X(s,t,z) = E(s, X(s,t,z)), \\ Z(t,t,z) = z, \end{cases} \tag{24}$$

with $z = (x,v)$ and $Z = (X,V)$. Then, the solution f can be written as $f(t,z) = f_0(Z(0,t,z))$ and we can get

$$\nabla_v f(t,z) = \nabla_v \left(f_0(Z(0,t,z))\right) \tag{25}$$

$$= \nabla_v X(0,t,z) \cdot (\nabla_x f_0)(Z(0,t,z)) + \nabla_v V(0,t,z) \cdot (\nabla_v f_0)(Z(0,t,z)). \tag{26}$$

Now, on the one hand we get

$$X(s,t,z) = x - \int_s^t V(r,t,z)\,dr = x - (t-s)\,v + \int_s^t (r-s)\,E(r, X(r,t,z))\,dr, \tag{27}$$

$$V(s,t,z) = v - \int_s^t E(r, X(r,t,z))\,dr. \tag{28}$$

Differentiating in v on both sides we get

$$\nabla_v X(s,t,z) = -(t-s)\,\mathrm{id} - \int_s^t (r-s)\,\nabla_v X(r,t,z)\,(\nabla_x E)(r, X(r,t,z))\,dr, \tag{29}$$

$$\nabla_v V(s,t,z) = \mathrm{id} - \int_s^t \nabla_v X(s,t,z)\,(\nabla_x E)(r, X(r,t,z))\,dr. \tag{30}$$

Given that $E \in L^\infty\left((0,T); C^{1,\alpha}\left(\mathbb{R}^3\right)\right)$ we get

$$\sup_z |\nabla_v X(s,t,z)| \leqslant t - s + T\,\|\nabla_x E\|_\infty \int_s^t \sup_z |\nabla_v X(r,t,z)|\,dr. \tag{31}$$

Using a Grömwall-like argument we get

$$\sup_z |\nabla_v X(s,t,z)| \leqslant (t-s)\left(1 + e^{T\|\nabla_x E\|_\infty (t-s)}\right) \leqslant T\left(1 + e^{T^2\|\nabla_x E\|_\infty}\right), \tag{32}$$

and therefore

$$\|\nabla_v X\|_\infty \leqslant T\left(1 + e^{T^2\|\nabla_x E\|_\infty}\right) \tag{33}$$

$$\|\nabla_v V\|_\infty \leqslant 1 + T^2\,\|\nabla_x E\|_\infty \left(1 + e^{T^2\|\nabla_x E\|_\infty}\right). \tag{34}$$

Moreover we get

$$|X(0,t,z) - (x - tv)| \leqslant \|E\|_\infty \frac{T^2}{2}, \qquad |V(0,t,z) - v| \leqslant \|E\|_\infty T. \tag{35}$$

Therefore, we get that $X(0,t,z) - tv \in B_{\frac{\|E\|_\infty}{2}T^2}(x)$ and $V(0,t,z) \in B_{\|E\|_\infty T}(v)$ and therefore there exists a constant $C > 0$ such that

$$
\begin{aligned}
|\nabla_v f(t,x,v)| &= |\nabla_v (f_0(X(0,t,z), V(0,t,z)))| && (36)\\
&\leqslant |\nabla_v X(0,t,z)(\nabla_x f_0)(Z(0,t,z))| && (37)\\
&\quad + |\nabla_v V(0,t,z)(\nabla_v f_0)(Z(0,t,z))| && (38)\\
&\leqslant C\,|(\nabla_v f_0)(X(0,t,z), V(0,t,z))| && (39)\\
&\leqslant C\,|(\nabla_v f_0)((X(0,t,z) - tv) + tv, V(0,t,z))| && (40)\\
&\leqslant C \operatorname*{ess\,sup}_{\substack{y\in B_R(x),\\ w\in B_R(v)}} |\nabla f_0(y+tv,w)|\,. && (41)
\end{aligned}
$$

This allows us to conclude.

□

Exercise 3

Let N be a positive integer number and consider ϕ a bounded, continuous and even potential and $\{(x_j, v_j)\}_{j=1}^N \subseteq \mathbb{R}^3 \times \mathbb{R}^3$. Define $\{(x_j(t), v_j(t))\}_{j=1}^N$ as the family of solutions of the Newton equations

$$
\begin{cases}
\partial_t x_j(t) = v_j(t)\,,\\
\partial_t v_j(t) = -\frac{1}{N}\sum_{k=1}^N \nabla\phi(x_j(t) - x_k(t))\,,\\
x_j(0) = x_j,\\
v_j(0) = v_j.
\end{cases}
\tag{42}
$$

Consider the measure μ_0 and μ_t given as[1]

$$
\mu_0(x,v) := \frac{1}{N}\sum_{j=1}^N \delta(x - x_j)\,\delta(v - v_j)\,, \tag{44}
$$

$$
\mu_t(x,v) := \frac{1}{N}\sum_{j=1}^N \delta(x - x_j(t))\,\delta(v - v_j(t))\,. \tag{45}
$$

Prove that μ_t is a weak solution of (1).

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. First of all we notice that

$$
\begin{aligned}
\langle \mu_t, \varphi\rangle : &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{N}\sum_{j=1}^N \varphi(x,v)\,\delta(x - x_j(t))\,\delta(v - v_j(t))\,dx\;dv && (46)\\
&= \frac{1}{N}\sum_{j=1}^N \varphi(x_j(t), v_j(t))\,. && (47)
\end{aligned}
$$

[1]Notice that the notation $\delta(x - x_0)$ indicates the delta at the point x_0 and justifies the notation

$$
\int_A \delta(x - x_0)\,dx = \begin{cases} 1, & x_0 \in A,\\ 0, & x_0 \notin A,\end{cases} \qquad\qquad \int_{\mathbb{R}^3} \varphi(x)\,\delta(x - x_0)\,dx = \varphi(x_0)\,. \tag{43}
$$

Given that x_j and v_j are continous for any $j = 1, \ldots, N$, we get that $t \mapsto \langle \mu_t, \varphi \rangle$ is continous. Moreover we get

$$\partial_t \langle \mu_t, \varphi \rangle := \frac{1}{N} \sum_{j=1}^{N} \partial_t \left[\varphi \left(x_j(t), v_j(t) \right) \right] \tag{48}$$

$$= \frac{1}{N} \sum_{j=1}^{N} \Bigg[v_j(t) \cdot \left(\nabla_x \varphi \right) \left(x_j(t), v_j(t) \right) \tag{49}$$

$$- \frac{1}{N} \sum_{k=1}^{N} \nabla \phi \left(x_j(t) - x_k(t) \right) \cdot \left(\nabla_v \varphi \right) \left(x_j(t), v_j(t) \right) \Bigg]. \tag{50}$$

Now, on the one hand we have

$$\langle v \cdot \nabla_x \mu_t, \varphi \rangle = -\langle \mu_t, \operatorname{div}_x (v \varphi) \rangle = -\langle \mu_t, v \cdot \nabla_x \varphi \rangle \tag{51}$$

$$= -\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{N} \sum_{j=1}^{N} v \cdot \nabla_x \varphi(x, v) \, \delta \left(x - x_j(t) \right) \delta \left(v - v_j(t) \right) dx \, dv \tag{52}$$

$$= -\frac{1}{N} \sum_{j=1}^{N} v_j(t) \cdot \left(\nabla_x \varphi \right) \left(x_j(t), v_j(t) \right), \tag{53}$$

which cancel the summation in (49). On the other hand, the density associated to μ_t is given by

$$\rho_t(x) = \int_{\mathbb{R}^3} d\mu_t(x, v) = \int_{\mathbb{R}^3} \frac{1}{N} \sum_{j=1}^{N} \delta \left(x - x_j(t) \right) \delta \left(v - v_j(t) \right) dv \tag{54}$$

$$= \frac{1}{N} \sum_{j=1}^{N} \delta \left(x - x_j(t) \right). \tag{55}$$

Therefore, we get that the electric field is of the form

$$E(t, x) = -\nabla \left(\phi * \rho_t \right)(x) = -\left((\nabla \phi) * \rho_t \right)(x) \tag{56}$$

$$= -\int_{\mathbb{R}^3} \frac{1}{N} \sum_{j=1}^{N} \nabla \phi (x - y) \, \delta \left(y - x_j(t) \right) dy \tag{57}$$

$$= -\frac{1}{N} \sum_{j=1}^{N} \nabla \phi \left(x - x_j(t) \right). \tag{58}$$

Finally, this implies

$$\langle E(t, \cdot) \cdot \nabla_v \mu_t, \varphi \rangle = -\langle \mu_t, \operatorname{div}_v \left(E(t, \cdot) \varphi \right) \rangle = -\langle \mu_t, E(t, \cdot) \cdot \nabla_v \varphi \rangle \tag{59}$$

$$= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{N} \sum_{k=1}^{N} \nabla \phi \left(x - x_k(t) \right) \right) \cdot \tag{60}$$

$$\cdot \nabla_v \varphi(x, v) \, \delta \left(x - x_j(t) \right) \delta \left(v - v_j(t) \right) dx \, dv \tag{61}$$

$$= \frac{1}{N^2} \sum_{1 \leqslant j, k \leqslant N} \nabla \phi \left(x_j(t) - x_k(t) \right) \cdot \left(\nabla_v \varphi \right) \left(x_j(t), v_j(t) \right). \tag{62}$$

This allows us to conclude that μ_t is a solution of

$$\partial_t \mu_t + v \cdot \nabla_x \mu_t + E \cdot \nabla_v \mu_t = 0. \tag{63}$$

□